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TOPICS IN COMPLEX ANALYSIS
FALL 2025
SHEET 4

Solution for 'Topics in complex analysis'

(01/10/2025)

H 4.1 (The construction of Mittag-Leffler)
Construct explicitly (i.e. as an explicit series) a holomorphic function $f : \mathbb{C}\setminus\{\sqrt{n} : n \in \mathbb{N}\} \to \mathbb{C}$ such that at $\sqrt{n}$ the function f has the principal part $q_n(z) = \frac{\sqrt{n}}{z-\sqrt{n}}$.
Hint: Prove that a second order Taylor polynomial p_n of q_n yields the local normal convergence of the sum $f = \sum_{n\in\mathbb{N}} q_n - p_n$ (cf. the statement of Theorem 2.2).

Solution H 4.1:
We follow the hint and first determine the Taylor series of each principal part. Using properties of the geometric series, for $|z| < \sqrt{n}$ we can write

$$\frac{\sqrt{n}}{z-\sqrt{n}} = \frac{-1}{1-\frac{z}{\sqrt{n}}} = -\sum_{j=0}^{\infty}\left(\frac{z}{\sqrt{n}}\right)^j.$$

According to the hint we choose as polynomials p_n in the Mittag-Leffler theorem (cf. Theorem 2.2 in the course) the second order Taylor-polynomials $p_n(z) = -1 - \frac{z}{\sqrt{n}} - \frac{z^2}{n}$. We have to estimate the difference $|q_n - p_n|$ on a fixed compact subset $K \subset \mathbb{C}\setminus\{\sqrt{n} : n \in \mathbb{N}\}$. A direct computation yields for $|z| < \sqrt{n}$ the estimate

$$\begin{aligned}|q_n(z) - p_n(z)| &\le \left|\sum_{j=3}^{\infty}\left(\frac{z}{\sqrt{n}}\right)^j\right| = \left|\sum_{j=0}^{\infty}\left(\frac{z}{\sqrt{n}}\right)^{j+3}\right| \le \left|\frac{z}{\sqrt{n}}\right|^3\sum_{j=0}^{\infty}\left|\frac{z}{\sqrt{n}}\right|^j \\ &= \left|\frac{z}{\sqrt{n}}\right|^3\frac{1}{1-\frac{|z|}{\sqrt{n}}}.\end{aligned}$$

Since the compact set K is bounded there exists $n(K)$ such that for all $n \ge n(K)$ we have $K \subset B_{\frac{\sqrt{n}}{2}}(0)$. Hence

$$\sum_{n\ge n(K)}\sup_{z\in K}|q_n(z) - p_n(z)| \le \sup_{z\in K}\frac{|z|^3}{1-\frac{1}{2}}\sum_{n\ge n(K)}\frac{1}{n^{3/2}} < +\infty.$$

In particular, this shows the local normal convergence of $f := \sum_{n=1}^{\infty} q_n - p_n$ on $\mathbb{C}\setminus\{\sqrt{n} : n \in \mathbb{N}\}$, so that f is holomorphic on $\mathbb{C}\setminus\{\sqrt{n} : n \in \mathbb{N}\}$. As seen in the lecture it follows that at each point $\sqrt{n}$ the principal part of the series is given by q_n. Finally, we obtain an explicit series

$$f(z) = \sum_{n=1}^{\infty}\left(\frac{z}{\sqrt{n}}\right)^3\frac{1}{1-\frac{z}{\sqrt{n}}} = \sum_{n=1}^{\infty}\frac{z^3}{n(\sqrt{n}-z)}.$$

Remark: The above function is an example for which the correcting polynomials p_n are indeed necessary. The sum of principal parts does not converge at any $z \in \mathbb{C}$.

H 4.2 (The partial fraction decomposition of $\frac{\pi^2}{\sin^2(\pi z)}$)
The goal of this exercise is to prove the formula

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}}\frac{1}{(z-n)^2}.$$

This will be achieved in several steps. Keep in mind that $\sin(z) = \frac{1}{2i}\left(\exp(iz) - \exp(-iz)\right)$.

a) Show that the function $f(z) := \frac{\pi^2}{\sin^2(\pi z)}$ has its singularities exactly at the points $z^* \in \mathbb{Z}$ and determine the principal parts of the Laurent series expansion in those points.

b) Show that the series $g(z) := \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$ converges locally uniformly on $\mathbb{C} \setminus \mathbb{Z}$, so that it is meromorphic. Conclude that the difference $g(z) - f(z)$ can be extended to an entire function.

c) Show that the function $z = x + iy \mapsto f(x + iy)$ vanishes when $|y| \to +\infty$ uniformly in x.

d) Show the same statement for the function g. Then prove that the difference $g - f$ is bounded on $\mathbb{C}$ and conclude the proof.
Hint: For d) it can be useful to note that $(g - f)(z + 1) = (g - f)(z)$ for all $z \in \mathbb{C}$.

Solution H 4.2:
a) The function f has singularities exactly at the zeros of $z \mapsto \sin(\pi z)$. As a standard fact we note that $\sin(\pi z) = 0$ for all $z \in \mathbb{Z}$. We argue that these are the only zeros. Let $z = x + iy$ with $x, y \in \mathbb{R}$. Then

$$\sin(z) = \frac{1}{2i}\left(e^{ix}e^{-y} - e^{-ix}e^{y}\right),$$

so that $\sin(z) = 0$ implies that $e^{2ix} = e^{2y}$. The left hand side has modulus 1, so that we conclude that $e^{2y} = 1$, which yields by injectivity of the real-valued exponential function that $y = 0$. On the other hand, the property $e^{2ix} \in (0, +\infty)$ implies that $x \in \pi\mathbb{Z}$.
In order to determine the principal part in a singularity $z^* \in \mathbb{Z}$ we first treat the case $z^* = 0$. Since the function $z \mapsto \frac{\sin(z)}{z}$ has a removable singularity at $z^* = 0$, it follows that f has a pole of second order at $z^* = 0$. Hence the Laurent series takes the form

$$f(z) = a_{-2}z^{-2} + a_{-1}z^{-1} + \sum_{n=0}^{\infty} a_n z^n.$$

Since the corresponding integrals for the coefficients are quite difficult to evaluate, we use this structure to determine the coefficients a_{-1} and a_{-2}. Note since $\lim_{z\to 0} \frac{\sin(z)}{z} = 1$ we have

$$a_{-2} = \lim_{z\to 0} z^2 f(z) = \lim_{z\to 0} \frac{(\pi z)^2}{\sin^2(\pi z)} = 1.$$

Next, for the coefficient a_{-1} we calculate

$$\begin{aligned} a_{-1} = \frac{\mathrm{d}}{\mathrm{d}z}(z^2 f(z))\Big|_{z=0} &= \lim_{z\to 0} \frac{2\pi^2 z \sin^2(\pi z) - 2\pi(\pi z)^2 \sin(\pi z)\cos(\pi z)}{\sin^4(\pi z)} \\ &= \lim_{z\to 0} \frac{2\pi - 2\pi\cos(\pi z)}{\sin(\pi z)} = 0, \end{aligned}$$

since $\cos(z) = 1 - z^2 + \mathcal{O}(z^4)$ as $z \to 0$. Hence the principal part at $z^* = 0$ reads $q(z) = \frac{1}{z^2}$. To treat the other singularities we note that $f(z + 1) = f(z)$ for all $z \in \mathbb{C} \setminus \mathbb{Z}$. Indeed, write $z = x + iy$ with $x, y \in \mathbb{R}$. Then

$$\sin(\pi z + \pi) = \frac{1}{2i}\left(e^{i\pi(x+1)}e^{-y} - e^{-i\pi(x+1)}e^{y}\right) = -\sin(\pi z),$$

so that taking the inverse square yields the claim. By this periodicity property it follows that the coefficients of the Laurent series are also periodic. Hence at a general $n \in \mathbb{Z}$ the principal part is given by $q_n(z) = \frac{1}{(z-n)^2}$.

b) Let $K \subset \mathbb{C}\setminus\mathbb{Z}$ be a compact set. Then there exists $c(K) < +\infty$ such that $\sup_{z\in K} |z| \le c(K)$. In particular, for all $|n| \ge 2c(K)$ and $z \in K$ we have $|z - n| \ge |n| - c(K) \ge \frac{1}{2}|n|$, which implies that

$$\sum_{|n|\geq 2c(K)} \sup_{z\in K} \left| \frac{1}{(z-n)^2} \right| \leq \sum_{|n|\geq 2c(K)} \frac{4}{n^2} < +\infty.$$

The terms $\frac{1}{(z-n)^2}$ with $|n| < 2c(K)$ are bounded in K (since they are holomorphic there), hence the series g converges locally normally on $\mathbb{C}\setminus\mathbb{Z}$. By Lemma 1.11 it follows that g is holomorphic on $\mathbb{C}\setminus\mathbb{Z}$. Since all singularities are isolated and poles of second order, we deduce that g is meromorphic on $\mathbb{C}$. Its principal parts agree with the principal parts of f at all singularities, so it follows from Remark 2.3 that the difference $g-f$ can be extended to an entire function.

c) We show that the function $|\sin(x+iy)|$ blows up when $|y| \to +\infty$ uniformly in $x \in \mathbb{R}$. Indeed, by the reverse triangle inequality we have

$$|\sin(x+iy)| = \left| \frac{1}{2i}\left(e^{ix}e^{-y} - e^{-ix}e^{y}\right) \right| \geq \frac{1}{2}\left(e^{|y|} - e^{-|y|}\right).$$

The right hand side goes to $+\infty$ when $|y| \to +\infty$ uniformly in $x \in \mathbb{R}$. This proves the claim.

d) Fix $n \in \mathbb{Z}$ and write $z = x+iy$ with $x, y \in \mathbb{R}$. By the periodicity $g(z+1) = g(z)$ for all $z \in \mathbb{C}\setminus\mathbb{Z}$ (which follows by an index shift) it suffices to take $x \in [0,1]$. Using the equivalence of the ℓ^1 and Euclidean norms on $\mathbb{R}^2$ (or just carefully computing with bare hands) we deduce that there exists a constant $c > 0$ such that

$$|z-n| \geq \frac{1}{c}(|y| + |x-n|) \geq \begin{cases} \frac{1}{c}(|y|+|n|-1) & \text{if } n \geq 1, \\ \frac{1}{c}(|y|+|n|) & \text{if } n \leq 0. \end{cases}$$

Hence, performing further index shifts, we infer that

$$\begin{aligned} \left| \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} \right| &\leq c^2 \sum_{n\geq 1} \frac{1}{(|y|+|n|-1)^2} + c^2 \sum_{n\leq 0} \frac{1}{(|y|+|n|)^2} \\ &\leq 2c^2 \sum_{n\geq 0} \frac{1}{(|y|+|n|)^2} \leq 2c^2 \sum_{m\geq |y|-1} \frac{1}{m^2}. \end{aligned}$$

Clearly the last term vanishes when $|y| \to +\infty$.
It remains to show that the difference $g-f$ is bounded on $\mathbb{C}$. We already know that it is holomorphic and periodic in the sense that $(g-f)(z+1) = (g-f)(z)$. Hence on each stripe of the form $\mathbb{R}\times[-a,a]$ it is bounded. From c) and the first part of d) we further know that there exists some $a^* > 0$ such that for all $z = x+iy$ with $x \in \mathbb{R}$ and $y \in \mathbb{R}$ with $|y| > a^*$ it holds that $|(g-f)(z)| \leq 1$. Hence $g-f$ is a bounded, entire function. By Liouville's theorem it is therefore constant and again by c) and the first part of d) it follows that $f = g$. □

H 4.3 (The Weierstrass elliptic function)
Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. Show that up to an additive constant, there exists exactly one holomorphic function $\wp : \mathbb{C}\setminus\{m\omega_1 + n\omega_2 : m,n \in \mathbb{Z}\} \to \mathbb{C}$ such that

(i) $\wp$ has principal part $q(z) = \frac{1}{z^2}$ at $d = 0$;

(ii) $\wp$ is $\{\omega_1,\omega_2\}$-periodic, i.e.

$$\wp(z+\omega_1) = \wp(z) \quad \text{and} \quad \wp(z+\omega_2) = \wp(z) \qquad \forall z \in \mathbb{C}\setminus\{m\omega_1+n\omega_2 : m,n\in\mathbb{Z}\}.$$

You can use without proof that $\displaystyle\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m^2+n^2)^{3/2}} < +\infty.$

Remark: If we require that the coefficient a_0 of the Laurent series at the origin vanishes, this function is called the Weierstrass $\wp$ function. This function $\wp$ and its derivative can be used to parametrize elliptic curves. The Laurent series coefficients of $\wp$ are called Eisenstein series, and are the simplest examples of modular forms.

Solution H 4.3:
We first show the uniqueness statement. Suppose that there are two functions $\wp_1, \wp_2 : \mathbb{C}\setminus\{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\} \to \mathbb{C}$ with the given properties (i) and (ii). Then the difference $g := \wp_1 - \wp_2$ has a removable singularity at 0, and by periodicity also at each point $z_{m,n} := m\omega_1 + n\omega_2$, where $m, n \in \mathbb{Z}$. We conclude that g can be extended to an entire function. Since the set $\{\omega_1, \omega_2\}$ forms a basis of $\mathbb{C}$ as a vector space over $\mathbb{R}$, the local boundedness of (the extended version of) g and the periodicity imply that g is bounded on $\mathbb{C}$. Hence g is constant by Liouville's theorem. This shows the claim of uniqueness (up to an additive constant).
Next we construct the function $\wp$. The basic idea is to use the structure given by the Mittag-Leffler theorem, so we consider the countable, closed, discrete set $S = \{z_{m,n} : m, n \in \mathbb{Z}\}$ and the corresponding principal parts $q_{m,n}(z) = \frac{1}{(z-z_{m,n})^2}$. The difficult part is how to choose the polynomials in order to ensure periodicity. Non-constant polynomials are never periodic, so we try to use constant ones. The 0^{th}-order Laurent series term of $q_{m,n}$ at 0 is given by $q_{m,n}(0) = \frac{1}{z_{m,n}^2}$ whenever $(m, n) \neq (0, 0)$, and by 0 if $(m, n) = (0, 0)$. Thus our ansatz is

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(z - z_{m,n})^2} - \frac{1}{z_{m,n}^2}.$$

We show that the series converges locally normally on $\mathbb{C}\setminus\{z_{m,n} : m, n \in \mathbb{Z}\}$. Let $K \subset \mathbb{C}\setminus\{z_{m,n} : m, n \in \mathbb{Z}\}$ be a compact set. Note that for all $z \in K$ we have

$$\left|\frac{1}{(z - z_{m,n})^2} - \frac{1}{z_{m,n}^2}\right| = \frac{|z^2 - 2z_{m,n}z|}{|z_{m,n}|^2|z - z_{m,n}|^2}. \tag{1}$$

Since K is bounded, we have that $C_K := \sup_{z\in K} |z| < +\infty$. Next note that the $\mathbb{R}$-linear mapping $\Omega : \mathbb{C} \to \mathbb{C}$ defined by $\Omega(1) = \omega_1$ and $\Omega(i) = \omega_2$ is invertible (as ω_1 and ω_2 are linearly independent over $\mathbb{R}$). Thus there exists a constant $c = c(\omega_1, \omega_2) > 0$ such that

$$|z_{m,n}| = |\Omega(m + ni)| \geq c|m + ni| = c\sqrt{m^2 + n^2}. \tag{2}$$

Hence there exists $n(K) \in \mathbb{N}$ such that for all $m, n \in \mathbb{Z}$ with $m^2 + n^2 \geq n(K)$ it holds that $|z_{m,n}| \geq 2C_K$. Note that for all $z \in K$ this implies

$$|z - z_{m,n}| \geq |z_{m,n}| - |z| \geq |z_{m,n}| - C_K \geq \frac{|z_{m,n}|}{2}. \tag{3}$$

Inserting this estimate combined with (2) into (1) yields that if $z \in K$ and $m^2 + n^2 \geq n(K)$ then there is a constant $C < +\infty$ such that

$$\left|\frac{1}{(z - z_{m,n})^2} - \frac{1}{z_{m,n}^2}\right| \leq \frac{|z|^2 + 2|z_{m,n}||z|}{|z_{m,n}|^2 \cdot \frac{|z_{m,n}|^2}{4}} \leq \frac{4C_K^2 + 8|z_{m,n}|C_K}{|z_{m,n}|^4} \leq \frac{10C_K}{|z_{m,n}|^3} \leq \frac{C}{(n^2 + m^2)^{3/2}}.$$

In fact $C := \frac{10C_K}{c^3}$ works above. From the hint (which can be shown for instance via comparison to an integral) we have

$$\sum_{\substack{m,n\in\mathbb{Z} \\ m^2+n^2\geq n(K)}} \frac{1}{(n^2 + m^2)^{3/2}} < +\infty.$$

Hence

$$\sum_{\substack{m,n\in\mathbb{Z}\\ n^2+m^2\geq n(K)}} \sup_{z\in K}\left|\frac{1}{(z-z_{m,n})^2}-\frac{1}{z_{m,n}^2}\right| < +\infty.$$

We conclude the local normal convergence of this series (since for the terms with $n^2+m^2 < n(K)$ we simply use that they are bounded in K), so $\wp$ is holomorphic. We can then use the familiar argument (manipulating the series freely due to the local normal convergence) to see that $\wp$ plainly satisfies (i), and the periodicity (ii) follows by an index shift. □